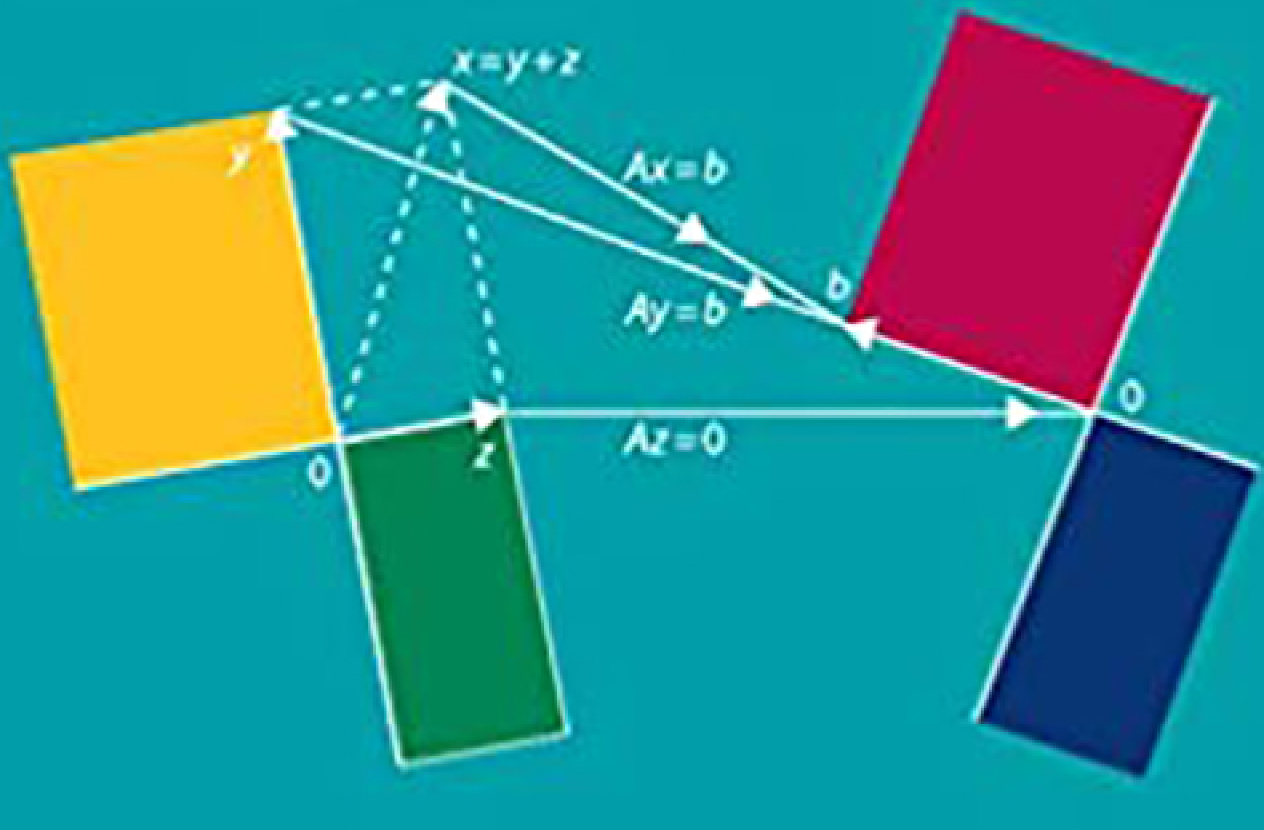
Introduction to
LINEAR ALGEBRA
FIFTH EDITION



GILBERT STRANG

INTRODUCTION TO LINEAR ALGEBRA

Fifth Edition

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The website for this book is **math.mit.edu/linearalgebra**.
The Solution Manual can be printed from that website.

Course material including syllabus and exams and also videotaped lectures are available on the book website and the teaching website: **web.mit.edu/18.06**

Linear Algebra is included in MIT's OpenCourseWare site **ocw.mit.edu**.
This provides video lectures of the full linear algebra course 18.06 and 18.06 SC.
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The front cover captures a central idea of linear algebra.
$A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the (red) column space of A.
One particular solution $\boldsymbol{y}$ is in the (yellow) row space: $A\boldsymbol{y} = \boldsymbol{b}$.
Add any vector $\boldsymbol{z}$ from the (green) nullspace of A: $A\boldsymbol{z} = \boldsymbol{0}$.
The complete solution is $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$. Then $A\boldsymbol{x} = A\boldsymbol{y} + A\boldsymbol{z} = \boldsymbol{b}$.

The cover design was the inspiration of Lois Sellers and Gail Corbett.

Table of Contents

Preface

I am happy for you to see this Fifth Edition of Introduction to Linear Algebra. This is the text for my video lectures on MIT's OpenCourseWare (**ocw.mit.edu** and also **YouTube**). I hope those lectures will be useful to you (maybe even enjoyable !).

Hundreds of colleges and universities have chosen this textbook for their basic linear algebra course. A sabbatical gave me a chance to prepare two new chapters about probability and statistics and understanding data. Thousands of other improvements too—probably only noticed by the author... Here is a new addition for students and all readers:

> Every section opens with a brief summary to explain its contents. When you read a new section, and when you revisit a section to review and organize it in your mind, those lines are a quick guide and an aid to memory.

Another big change comes on this book's website **math.mit.edu/linearalgebra**. That site now contains solutions to the Problem Sets in the book. With unlimited space, this is much more flexible than printing short solutions. There are three key websites :

ocw.mit.edu Messages come from thousands of students and faculty about linear algebra on this OpenCourseWare site. The 18.06 and 18.06 SC courses include video lectures of a complete semester of classes. Those lectures offer an independent review of the whole subject based on this textbook—the professor's time stays free and the student's time can be 2 a.m. (The reader doesn't have to be in a class at all.) Six million viewers around the world have seen these videos (*amazing*). I hope you find them helpful.

web.mit.edu/18.06 This site has homeworks and exams (with solutions) for the current course as it is taught, and as far back as 1996. There are also review questions, Java demos, Teaching Codes, and short essays (*and the video lectures*). My goal is to make this book as useful to you as possible, with all the course material we can provide.

math.mit.edu/linearalgebra This has become an active website. It now has Solutions to Exercises—with space to explain ideas. There are also new exercises from many different sources—practice problems, development of textbook examples, codes in MATLAB and *Julia* and *Python*, plus whole collections of exams (18.06 and others) for review. Please visit this linear algebra site. *Send suggestions to* **linearalgebrabook@gmail.com**

The Fifth Edition

The cover shows the **Four Fundamental Subspaces**—the row space and nullspace are on the left side, the column space and the nullspace of A^{T} are on the right. It is not usual to put the central ideas of the subject on display like this! When you meet those four spaces in Chapter 3, you will understand why that picture is so central to linear algebra.

Those were named the Four Fundamental Subspaces in my first book, and they start from a matrix A. Each row of A is a vector in n-dimensional space. When the matrix has m rows, each column is a vector in m-dimensional space. The crucial operation in linear algebra is to take ***linear combinations of column vectors***. This is exactly the result of a matrix-vector multiplication. *Ax is a combination of the columns of A.*

When we take *all* combinations Ax of the column vectors, we get the *column space.* If this space includes the vector b, we can solve the equation $Ax = b$.

May I call special attention to Section 1.3, where these ideas come early—with two specific examples. You are not expected to catch every detail of vector spaces in one day! But you will see the first matrices in the book, and a picture of their column spaces. There is even an *inverse matrix* and its connection to calculus. You will be learning the language of linear algebra in the best and most efficient way: by using it.

Every section of the basic course ends with a large collection of review problems. They ask you to use the ideas in that section—-the dimension of the column space, a basis for that space, the rank and inverse and determinant and eigenvalues of A. Many problems look for computations by hand on a small matrix, and they have been highly praised. The *Challenge Problems* go a step further, and sometimes deeper. Let me give four examples:

Section 2.1: Which row exchanges of a Sudoku matrix produce another Sudoku matrix?

Section 2.7: If P is a permutation matrix, why is some power P^k equal to I ?

Section 3.4: If $Ax = b$ and $Cx = b$ have the same solutions for every b, does A equal C ?

Section 4.1: What conditions on the four vectors r, n, c, ℓ allow them to be bases for the row space, the nullspace, the column space, and the left nullspace of a 2 by 2 matrix?

The Start of the Course

The equation $Ax = b$ uses the language of linear combinations right away. The vector Ax is *a combination of the columns of A*. The equation is asking for *a combination that produces b*. The solution vector x comes at three levels and all are important:

1. ***Direct solution*** to find x by forward elimination and back substitution.

2. ***Matrix solution*** using the inverse matrix: $x = A^{-1}b$ (if A has an inverse).

3. ***Particular solution*** (to $Ay = b$) plus ***nullspace solution*** (to $Az = 0$).

That vector space solution $x = y + z$ is shown on the cover of the book.

Direct elimination is the most frequently used algorithm in scientific computing. The matrix A becomes triangular—then solutions come quickly. We also see bases for the four subspaces. But don't spend forever on practicing elimination ... good ideas are coming.

The speed of every new supercomputer is tested on $Ax = b$: pure linear algebra. But even a supercomputer doesn't want the inverse matrix: *too slow*. Inverses give the simplest formula $x = A^{-1}b$ but not the top speed. And everyone must know that determinants are even slower—there is no way a linear algebra course should begin with formulas for the determinant of an n by n matrix. Those formulas have a place, but not first place.

Structure of the Textbook

Already in this preface, you can see the style of the book and its goal. That goal is serious, to explain this beautiful and useful part of mathematics. You will see how the applications of linear algebra reinforce the key ideas. This book moves gradually and steadily from *numbers* to *vectors* to *subspaces*—each level comes naturally and everyone can get it.

Here are 12 points about learning and teaching from this book:

1. Chapter 1 starts with vectors and dot products. If the class has met them before, focus quickly on linear combinations. Section 1.3 provides three independent vectors whose combinations fill all of 3-dimensional space, and three dependent vectors in a plane. ***Those two examples are the beginning of linear algebra***.

2. Chapter 2 shows the row picture and the column picture of $Ax = b$. The heart of linear algebra is in that connection between the rows of A and the columns of A: the same numbers but very different pictures. Then begins the algebra of matrices: an elimination matrix E multiplies A to produce a zero. The goal is to capture the whole process—start with A, multiply by E's, end with U.

   Elimination is seen in the beautiful form $A = LU$. The ***lower triangular*** L holds the forward elimination steps, and U is ***upper triangular*** for back substitution.

3. Chapter 3 is linear algebra at the best level: ***subspaces***. The column space contains all linear combinations of the columns. The crucial question is: ***How many of those columns are needed***? The answer tells us the dimension of the column space, and the key information about A. We reach the Fundamental Theorem of Linear Algebra.

4. With more equations than unknowns, it is almost sure that $Ax = b$ has no solution. We cannot throw out every measurement that is close but not perfectly exact! When we solve by ***least squares***, the key will be the matrix $A^{\mathrm{T}}A$. This wonderful matrix appears everywhere in applied mathematics, when A is rectangular.

5. ***Determinants*** give formulas for all that has come before—Cramer's Rule, inverse matrices, volumes in n dimensions. We don't need those formulas to compute. They slow us down. But $\det A = 0$ tells when a matrix is singular: this is the key to eigenvalues.

6. ***Section 6.1 explains eigenvalues for 2 by 2 matrices.*** Many courses want to see eigenvalues early. It is completely reasonable to come here directly from Chapter 3, because the determinant is easy for a 2 by 2 matrix. *The key equation is* $Ax = \lambda x$.

   Eigenvalues and eigenvectors are an astonishing way to understand a square matrix. They are not for $Ax = b$, they are for dynamic equations like $du/dt = Au$. The idea is always the same: *follow the eigenvectors*. In those special directions, A acts like a single number (the eigenvalue λ) and the problem is one-dimensional.

   An essential highlight of Chapter 6 is ***diagonalizing a symmetric matrix***. When all the eigenvalues are positive, the matrix is "positive definite". This key idea connects the whole course—positive pivots and determinants and eigenvalues and energy. I work hard to reach this point in the book and to explain it by examples.

7. Chapter 7 is new. It introduces ***singular values*** and ***singular vectors***. They separate all martices into simple pieces, ranked in order of their importance. You will see one way to compress an image. Especially you can analyze a matrix full of data.

8. Chapter 8 explains ***linear transformations***. This is geometry without axes, algebra with no coordinates. When we choose a basis, we reach the best possible matrix.

9. Chapter 9 moves from real numbers and vectors to complex vectors and matrices. The Fourier matrix F is the most important complex matrix we will ever see. And the ***Fast Fourier Transform*** (multiplying quickly by F and F^{-1}) is revolutionary.

10. Chapter 10 is full of applications, more than any single course could need:

    10.1 *Graphs and Networks*—leading to the edge-node matrix for Kirchhoff's Laws

    10.2 *Matrices in Engineering*—differential equations parallel to matrix equations

    10.3 *Markov Matrices*—as in Google's *PageRank* algorithm

    10.4 *Linear Programming*—a new requirement $x \geq 0$ and minimization of the cost

    10.5 *Fourier Series*—linear algebra for functions and digital signal processing

    10.6 *Computer Graphics*—matrices move and rotate and compress images

    10.7 *Linear Algebra in Cryptography*—this new section was fun to write. The Hill Cipher is not too secure. It uses modular arithmetic: integers from 0 to $p - 1$. Multiplication gives $4 \times 5 \equiv 1\,(mod\ 19)$. For decoding this gives $4^{-1} \equiv 5$.

11. How should computing be included in a linear algebra course? It can open a new understanding of matrices—every class will find a balance. MATLAB and *Maple* and *Mathematica* are powerful in different ways. *Julia* and *Python* are free and directly accessible on the Web. Those newer languages are powerful too !

    Basic commands begin in Chapter 2. Then Chapter 11 moves toward professional algorithms.You can upload and download codes for this course on the website.

12. Chapter 12 on Probability and Statistics is new, with truly important applications. When random variables are not independent we get covariance matrices. Fortunately they are symmetric positive definite. The linear algebra in Chapter 6 is needed now.

The Variety of Linear Algebra

Calculus is mostly about one special operation (the derivative) and its inverse (the integral). Of course I admit that calculus could be important . . . . But so many applications of mathematics are discrete rather than continuous, digital rather than analog. The century of data has begun! You will find a light-hearted essay called "Too Much Calculus" on my website. ***The truth is that vectors and matrices have become the language to know.***

Part of that language is the wonderful variety of matrices. Let me give three examples:

Symmetric matrix	*Orthogonal matrix*	*Triangular matrix*
$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A key goal is learning to "read" a matrix. You need to see the meaning in the numbers. This is really the essence of mathematics—patterns and their meaning.

I have used *italics* and ***boldface*** to pick out the key words on each page. I know there are times when you want to read quickly, looking for the important lines.

May I end with this thought for professors. You might feel that the direction is right, and wonder if your students are ready. ***Just give them a chance***! Literally thousands of students have written to me, frequently with suggestions and surprisingly often with thanks. They know this course has a purpose, because the professor and the book are on their side. Linear algebra is a fantastic subject, enjoy it.

Help With This Book

The greatest encouragement of all is the feeling that you are doing something worthwhile with your life. Hundreds of generous readers have sent ideas and examples and corrections (and favorite matrices) that appear in this book. *Thank you all.*

One person has helped with every word in this book. He is Ashley C. Fernandes, who prepared the LaTeX files. It is now six books that he has allowed me to write and rewrite, aiming for accuracy and also for life. Working with friends is a happy way to live.

Friends inside and outside the MIT math department have been wonderful. Alan Edelman for *Julia* and much more, Alex Townsend for the flag examples in 7.1, and Peter Kempthorne for the finance example in 7.3 : those stand out. Don Spickler's website on cryptography is simply excellent. I thank Jon Bloom, Jack Dongarra, Hilary Finucane, Pavel Grinfeld, Randy LeVeque, David Vogan, Liang Wang, and Karen Willcox. The "eigenfaces" in 7.3 came from Matthew Turk and Jeff Jauregui. And the big step to singular values was accelerated by Raj Rao's great course at Michigan.

This book owes so much to my happy sabbatical in Oxford. Thank you, Nick Trefethen and everyone. Especially you the reader! Best wishes in your work.

Background of the Author

This is my 9th textbook on linear algebra, and I hesitate to write about myself. It is the mathematics that is important, and the reader. The next paragraphs add something brief and personal, as a way to say that textbooks are written by people.

I was born in Chicago and went to school in Washington and Cincinnati and St. Louis. My college was MIT (and my linear algebra course was *extremely abstract*). After that came Oxford and UCLA, then back to MIT for a very long time. I don't know how many thousands of students have taken 18.06 (more than 6 million when you include the videos on ***ocw.mit.edu***). The time for a fresh approach was right, because this fantastic subject was only revealed to math majors—**we needed to open linear algebra to the world.**

I am so grateful for a life of teaching mathematics, more than I could possibly tell you.

Gilbert Strang

PS I hope the next book (2018 ?) will include *Learning from Data*. This subject is growing quickly, especially "deep learning". By knowing a function on a training set of old data, we approximate the function on new data. The approximation only uses one simple nonlinear function $f(x) = \max(0, x)$. It is n matrix multiplications that we optimize to make the learning deep: $\boldsymbol{x}_1 = f(A_1\boldsymbol{x} + \boldsymbol{b}_1), \boldsymbol{x}_2 = f(A_2\boldsymbol{x}_1 + \boldsymbol{b}_2), \ldots, \boldsymbol{x}_n = f(A_n\boldsymbol{x}_{n-1} + \boldsymbol{b}_n)$. Those are $n-1$ hidden layers between the input $\boldsymbol{x}$ and the output $\boldsymbol{x}_n$—which approximates $F(\boldsymbol{x})$ on the training set.

THE MATRIX ALPHABET

A	Any Matrix	P	Permutation Matrix
B	Basis Matrix	P	Projection Matrix
C	Cofactor Matrix	Q	Orthogonal Matrix
D	Diagonal Matrix	R	Upper Triangular Matrix
E	Elimination Matrix	R	Reduced Echelon Matrix
F	Fourier Matrix	S	Symmetric Matrix
H	Hadamard Matrix	T	Linear Transformation
I	Identity Matrix	U	Upper Triangular Matrix
J	Jordan Matrix	U	Left Singular Vectors
K	Stiffness Matrix	V	Right Singular Vectors
L	Lower Triangular Matrix	X	Eigenvector Matrix
M	Markov Matrix	Λ	Eigenvalue Matrix
N	Nullspace Matrix	Σ	Singular Value Matrix

Chapter 1

Introduction to Vectors

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $\boldsymbol{v}+\boldsymbol{w}$. We multiply them by numbers c and d to get $c\boldsymbol{v}$ and $d\boldsymbol{w}$. Combining those two operations (adding $c\boldsymbol{v}$ to $d\boldsymbol{w}$) gives the ***linear combination*** $c\boldsymbol{v}+d\boldsymbol{w}$.

Linear combination $$c\boldsymbol{v}+d\boldsymbol{w}=c\begin{bmatrix}1\\1\end{bmatrix}+d\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}c+2d\\c+3d\end{bmatrix}$$

Example $\boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$ is the combination with $c=d=1$

Linear combinations are all-important in this subject! Sometimes we want one particular combination, the specific choice $c=2$ and $d=1$ that produces $c\boldsymbol{v}+d\boldsymbol{w}=(4,5)$. Other times we want *all the combinations* of $\boldsymbol{v}$ and $\boldsymbol{w}$ (coming from all c and d).

The vectors $c\boldsymbol{v}$ lie along a line. When $\boldsymbol{w}$ is not on that line, **the combinations** $c\boldsymbol{v}+d\boldsymbol{w}$ **fill the whole two-dimensional plane**. Starting from four vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}$ in four-dimensional space, their combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}+f\boldsymbol{z}$ are likely to fill the space—but not always. The vectors and their combinations could lie in a plane or on a line.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n-dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n-dimensional space). The first steps are the operations in Sections 1.1 and 1.2. Then Section 1.3 outlines three fundamental ideas.

1.1 ***Vector addition*** $\boldsymbol{v}+\boldsymbol{w}$ ***and linear combinations*** $c\boldsymbol{v}+d\boldsymbol{w}$.

1.2 ***The dot product*** $\boldsymbol{v}\cdot\boldsymbol{w}$ ***of two vectors and the length*** $\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$.

1.3 ***Matrices*** A, ***linear equations*** $A\boldsymbol{x}=\boldsymbol{b}$, ***solutions*** $\boldsymbol{x}=A^{-1}\boldsymbol{b}$.

1.1 Vectors and Linear Combinations

1 $3\boldsymbol{v} + 5\boldsymbol{w}$ is a typical **linear combination** $c\boldsymbol{v} + d\boldsymbol{w}$ of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

2 For $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ that combination is $3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+10 \\ 3+15 \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$.

3 The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ goes across to $x = 2$ and up to $y = 3$ in the xy plane.

4 The combinations $c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ fill the whole xy plane. They produce every $\begin{bmatrix} x \\ y \end{bmatrix}$.

5 The combinations $c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ fill a **plane** in xyz space. Same plane for $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

6 But $\begin{matrix} c + 2d = 1 \\ c + 3d = 0 \\ c + 4d = 0 \end{matrix}$ has no solution because its right side $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not on that plane.

"You can't add apples and oranges." In a strange way, this is the reason for vectors. We have two separate numbers v_1 and v_2. That pair produces a ***two-dimensional vector*** $\boldsymbol{v}$:

$$\textbf{Column vector } \boldsymbol{v} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{aligned} v_1 &= \text{first component of } \boldsymbol{v} \\ v_2 &= \text{second component of } \boldsymbol{v} \end{aligned}$$

We write $\boldsymbol{v}$ as a ***column***, not as a row. The main point so far is to have a single letter $\boldsymbol{v}$ (in ***boldface italic***) for this pair of numbers v_1 and v_2 (in *lightface italic*).

Even if we don't add v_1 to v_2, we do ***add vectors***. The first components of $\boldsymbol{v}$ and $\boldsymbol{w}$ stay separate from the second components:

$$\textbf{VECTOR ADDITION} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

Subtraction follows the same idea: *The components of* $\boldsymbol{v} - \boldsymbol{w}$ *are* $v_1 - w_1$ *and* $v_2 - w_2$.

The other basic operation is *scalar multiplication*. Vectors can be multiplied by 2 or by -1 or by any number c. To find $2\boldsymbol{v}$, multiply each component of $\boldsymbol{v}$ by 2:

$$\textbf{SCALAR MULTIPLICATION} \qquad 2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \boldsymbol{v} + \boldsymbol{v} \qquad -\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

The components of $c\boldsymbol{v}$ are cv_1 and cv_2. The number c is called a "scalar".

Notice that the sum of $-\boldsymbol{v}$ and $\boldsymbol{v}$ is the zero vector. This is $\mathbf{0}$, which is not the same as the number zero! The vector $\mathbf{0}$ has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$ and $d\boldsymbol{w}$—***adding vectors and multiplying by scalars***.

Linear Combinations

Now we combine addition with scalar multiplication to produce a "**linear combination**" of $\boldsymbol{v}$ and $\boldsymbol{w}$. Multiply $\boldsymbol{v}$ by c and multiply $\boldsymbol{w}$ by d. Then add $c\boldsymbol{v} + d\boldsymbol{w}$.

The sum of $c\boldsymbol{v}$ ***and*** $d\boldsymbol{w}$ ***is a*** *linear combination* $c\boldsymbol{v} + d\boldsymbol{w}$.

Four special linear combinations are: sum, difference, zero, and a scalar multiple $c\boldsymbol{v}$:

$$\begin{aligned} 1\boldsymbol{v} + 1\boldsymbol{w} &= \text{sum of vectors in Figure 1.1a} \\ 1\boldsymbol{v} - 1\boldsymbol{w} &= \text{difference of vectors in Figure 1.1b} \\ 0\boldsymbol{v} + 0\boldsymbol{w} &= \textbf{\textit{zero vector}} \\ c\boldsymbol{v} + 0\boldsymbol{w} &= \text{vector } c\boldsymbol{v} \text{ in the direction of } \boldsymbol{v} \end{aligned}$$

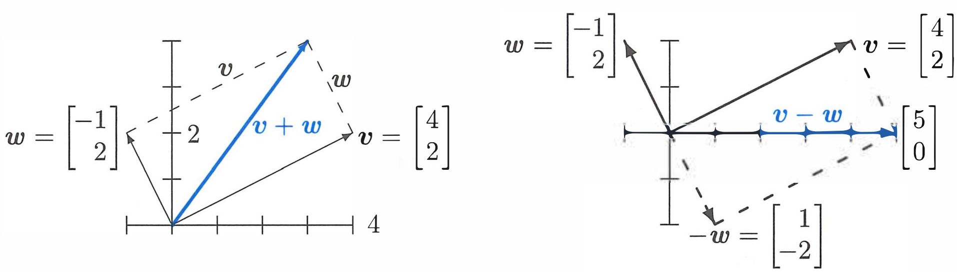
The zero vector is always a possible combination (its coefficients are zero). Every time we see a "space" of vectors, that zero vector will be included. This big view, taking *all* the combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$, is linear algebra at work.

The figures show how you can visualize vectors. For algebra, we just need the components (like 4 and 2). That vector $\boldsymbol{v}$ is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up. It ends at the point whose x, y coordinates are $4, 2$. This point is another representation of the vector—so we have three ways to describe $\boldsymbol{v}$:

Represent vector $\boldsymbol{v}$ | Two numbers | Arrow from $(0, 0)$ | Point in the plane

We add using the numbers. We visualize $\boldsymbol{v} + \boldsymbol{w}$ using arrows:

Vector addition (head to tail) ***At the end of*** $\boldsymbol{v}$, ***place the start of*** $\boldsymbol{w}$.



$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Figure 1.1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The reverse of $\boldsymbol{w}$ is $-\boldsymbol{w}$. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

We travel along $\boldsymbol{v}$ and then along $\boldsymbol{w}$. Or we take the diagonal shortcut along $\boldsymbol{v} + \boldsymbol{w}$. We could also go along $\boldsymbol{w}$ and then $\boldsymbol{v}$. In other words, $\boldsymbol{w} + \boldsymbol{v}$ **gives the same answer as** $\boldsymbol{v} + \boldsymbol{w}$. These are different ways along the parallelogram (in this example it is a rectangle).

Vectors in Three Dimensions

A vector with two components corresponds to a point in the xy plane. The components of $\boldsymbol{v}$ are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2), when it starts from $(0,0)$. Now we allow vectors to have three components (v_1, v_2, v_3).

The xy plane is replaced by three-dimensional xyz space. Here are typical vectors (still column vectors but with three components):

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}+\boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

The vector $\boldsymbol{v}$ corresponds to an arrow in 3-space. Usually the arrow starts at the "origin", where the xyz axes meet and the coordinates are $(0,0,0)$. The arrow ends at the point with coordinates v_1, v_2, v_3. There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends***.

The vector (x, y) in the plane is different from $(x, y, 0)$ in 3-space!

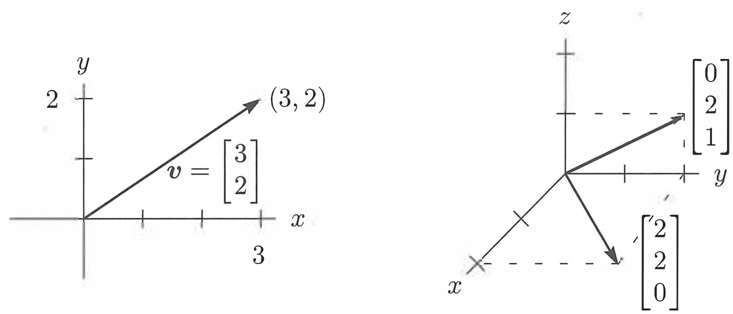


Figure 1.2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

From now on $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ***is also written as*** $\boldsymbol{v} = (1, 1, -1)$.

The reason for the row form (in parentheses) is to save space. But $\boldsymbol{v} = (1, 1, -1)$ is not a row vector! It is in actuality a column vector, just temporarily lying down. The row vector $[1 \;\; 1 \;\; -1]$ is absolutely different, even though it has the same three components. That 1 by 3 row vector is the "transpose" of the 3 by 1 column vector $\boldsymbol{v}$.

In three dimensions, $\boldsymbol{v} + \boldsymbol{w}$ is still found a component at a time. The sum has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$. You see how to add vectors in 4 or 5 or n dimensions. When $\boldsymbol{w}$ starts at the end of $\boldsymbol{v}$, the third side is $\boldsymbol{v} + \boldsymbol{w}$. The other way around the parallelogram is $\boldsymbol{w} + \boldsymbol{v}$. Question: Do the four sides all lie in the same plane? *Yes*. And the sum $\boldsymbol{v} + \boldsymbol{w} - \boldsymbol{v} - \boldsymbol{w}$ goes completely around to produce the _____ vector.

A typical linear combination of three vectors in three dimensions is $\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w}$:

Linear combination
Multiply by $1, 4, -2$
Then add

$$\begin{bmatrix}1\\0\\3\end{bmatrix} + 4\begin{bmatrix}1\\2\\1\end{bmatrix} - 2\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\9\end{bmatrix}.$$

The Important Questions

For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$. For two vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v}$. For three vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$. Will you take the big step from *one* combination to **all combinations**? Every c and d and e are allowed. Suppose the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in three-dimensional space:

1. What is the picture of *all* combinations $c\boldsymbol{u}$?
2. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v}$?
3. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

The answers depend on the particular vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. If they were zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

1. The combinations $c\boldsymbol{u}$ fill a ***line through*** $(\mathbf{0}, \mathbf{0}, \mathbf{0})$.
2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane through*** $(\mathbf{0}, \mathbf{0}, \mathbf{0})$.
3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

The zero vector $(0, 0, 0)$ is on the line because c can be zero. It is on the plane because c and d could both be zero. The line of vectors $c\boldsymbol{u}$ is infinitely long (forward and backward). It is the plane of all $c\boldsymbol{u} + d\boldsymbol{v}$ (combining two vectors in three-dimensional space) that I especially ask you to think about.

Adding all $c\boldsymbol{u}$ on one line to all $d\boldsymbol{v}$ on the other line fills in the plane in Figure 1.3.

When we include a third vector $\boldsymbol{w}$, the multiples $e\boldsymbol{w}$ give a third line. **Suppose that third line is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.** Then combining all $e\boldsymbol{w}$ with all $c\boldsymbol{u} + d\boldsymbol{v}$ fills up the whole three-dimensional space.

This is the typical situation! **Line**, then **plane**, then **space**. But other possibilities exist. When $\boldsymbol{w}$ happens to be $c\boldsymbol{u} + d\boldsymbol{v}$, that third vector $\boldsymbol{w}$ is in the plane of the first two. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ will not go outside that $\boldsymbol{uv}$ plane. We do not get the full three-dimensional space. Please think about the special cases in Problem 1.

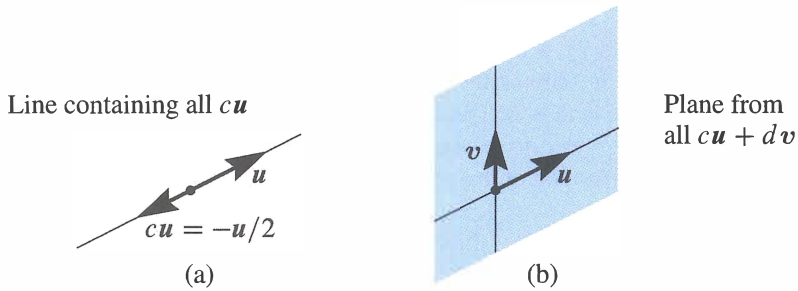


Figure 1.3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

■ REVIEW OF THE KEY IDEAS ■

1. A vector $\boldsymbol{v}$ in two-dimensional space has two components v_1 and v_2.
2. $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1,\, v_2 + w_2)$ and $c\boldsymbol{v} = (cv_1, cv_2)$ are found a component at a time.
3. A linear combination of three vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ is $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$.
4. Take *all* linear combinations of $\boldsymbol{u}$, or $\boldsymbol{u}$ and $\boldsymbol{v}$, or $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. In three dimensions, those combinations typically fill a line, then a plane, then the whole space $\mathbf{R}^3$.

■ WORKED EXAMPLES ■

1.1 A The linear combinations of $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (0, 1, 1)$ fill a plane in $\mathbf{R}^3$. *Describe that plane*. Find a vector that is *not* a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$—not on the plane.

Solution The plane of $\boldsymbol{v}$ and $\boldsymbol{w}$ contains all combinations $c\boldsymbol{v} + d\boldsymbol{w}$. The vectors in that plane allow any c and d. The plane of Figure 1.3 fills in between the two lines.

$$\text{Combinations} \quad c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix} \text{ fill a plane.}$$

Four vectors in that plane are $(0, 0, 0)$ and $(2, 3, 1)$ and $(5, 7, 2)$ and $(\pi, 2\pi, \pi)$. The second component $c + d$ is always the sum of the first and third components. Like most vectors, $(1, 2, 3)$ *is* ***not*** *in the plane, because* $2 \neq 1 + 3$.

Another description of this plane through $(0, 0, 0)$ is to know that $\boldsymbol{n} = (1, -1, 1)$ is **perpendicular** to the plane. Section 1.2 will confirm that 90° angle by testing dot products: $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n} = 0$. Perpendicular vectors have zero dot products.

1.1 B For $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$, describe all points $c\boldsymbol{v}$ with (**1**) *whole numbers* c (**2**) *nonnegative* numbers $c \geq 0$. Then add all vectors $d\boldsymbol{w}$ and describe all $c\boldsymbol{v} + d\boldsymbol{w}$.

Solution

(**1**) The vectors $c\boldsymbol{v} = (c, 0)$ with whole numbers c are **equally spaced points** along the x axis (the direction of $\boldsymbol{v}$). They include $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$.

(**2**) The vectors $c\boldsymbol{v}$ with $c \geq 0$ fill a ***half-line***. It is the positive x axis. This half-line starts at $(0, 0)$ where $c = 0$. It includes $(100, 0)$ and $(\pi, 0)$ but not $(-100, 0)$.

(**1′**) Adding all vectors $d\boldsymbol{w} = (0, d)$ puts a vertical line through those equally spaced $c\boldsymbol{v}$. We have infinitely many ***parallel lines*** from (*whole number c, any number d*).

(**2′**) Adding all vectors $d\boldsymbol{w}$ puts a vertical line through every $c\boldsymbol{v}$ on the half-line. Now we have a ***half-plane***. The right half of the xy plane has any $x \geq 0$ and any y.

1.1 C Find two equations for c and d so that **the linear combination** $c\boldsymbol{v} + d\boldsymbol{w}$ **equals** $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution In applying mathematics, many problems have two parts:

1 *Modeling part* Express the problem by a set of equations.

2 *Computational part* Solve those equations by a fast and accurate algorithm.

Here we are only asked for the first part (the equations). Chapter 2 is devoted to the second part (the solution). Our example fits into a fundamental model for linear algebra:

$$\text{Find } n \text{ numbers } \quad c_1, \ldots, c_n \quad \text{so that} \quad c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{b}.$$

For $n = 2$ we will find a formula for the $\boldsymbol{c}$'s. The "elimination method" in Chapter 2 succeeds far beyond $n = 1000$. For n greater than 1 billion, see Chapter 11. Here $n = 2$:

Vector equation
$c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b}$

$$c \begin{bmatrix} 2 \\ -1 \end{bmatrix} + d \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The required equations for c and d just come from the two components separately:

Two ordinary equations

$$\begin{aligned} 2c - d &= 1 \\ -c + 2d &= 0 \end{aligned}$$

Each equation produces a line. The two lines cross at the solution $c = \frac{2}{3}, d = \frac{1}{3}$. Why not see this also as a **matrix equation**, since that is where we are going:

2 by 2 matrix

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1 Describe geometrically (line, plane, or all of $\mathbf{R}^3$) all linear combinations of

$$\text{(a)} \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } \begin{bmatrix}3\\6\\9\end{bmatrix} \qquad \text{(b)} \begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\2\\3\end{bmatrix} \qquad \text{(c)} \begin{bmatrix}2\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\2\\2\end{bmatrix} \text{ and } \begin{bmatrix}2\\2\\3\end{bmatrix}$$

2 Draw $\boldsymbol{v} = \begin{bmatrix}4\\1\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}-2\\2\end{bmatrix}$ and $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ in a single xy plane.

3 If $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix}5\\1\end{bmatrix}$ and $\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix}1\\5\end{bmatrix}$, compute and draw the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

4 From $\boldsymbol{v} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}1\\2\end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

5 Compute $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$ and $2\boldsymbol{u} + 2\boldsymbol{v} + \boldsymbol{w}$. How do you know $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie in a plane?

These lie in a plane because $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. Find c and d

$$\boldsymbol{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix}-3\\1\\-2\end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix}2\\-3\\-1\end{bmatrix}.$$

6 Every combination of $\boldsymbol{v} = (1, -2, 1)$ and $\boldsymbol{w} = (0, 1, -1)$ has components that add to ______. Find c and d so that $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, -6)$. Why is $(3, 3, 6)$ impossible?

7 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix}2\\1\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

8 The parallelogram in Figure 1.1 has diagonal $\boldsymbol{v} + \boldsymbol{w}$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.

9 If three corners of a parallelogram are $(1, 1)$, $(4, 2)$, and $(1, 3)$, what are all three of the possible fourth corners? Draw two of them.

Problems 10–14 are about special vectors on cubes and clocks in Figure 1.4.

10 Which point of the cube is $\boldsymbol{i} + \boldsymbol{j}$? Which point is the vector sum of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$? Describe all points (x, y, z) in the cube.

11 Four corners of this unit cube are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces are ______. The cube has how many edges?

12 *Review Question.* In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$?

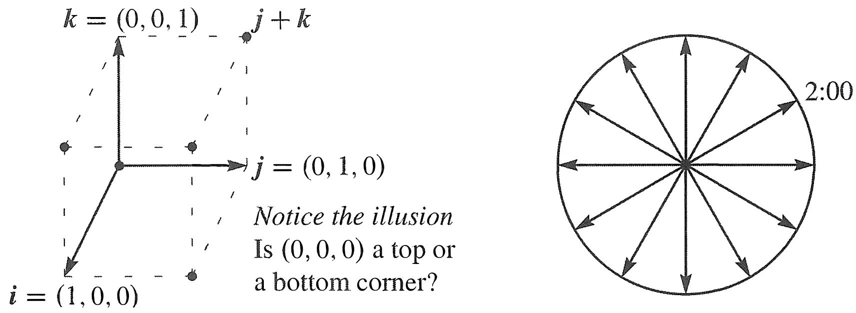


Figure 1.4: Unit cube from $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and twelve clock vectors.

13 (a) What is the sum V of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

(c) What are the x, y components of that 2:00 vector $\boldsymbol{v} = (\cos\theta, \sin\theta)$?

14 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0, 0)$ at the center. The vector to 12:00 is doubled to $(0, 2)$. The new twelve vectors add to ____.

Problems 15–19 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (Figure 1.5a).

15 Figure 1.5a shows $\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$.

16 Mark the point $-\boldsymbol{v} + 2\boldsymbol{w}$ and any other combination $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$.

17 Locate $\frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{2}{3}\boldsymbol{v} + \frac{2}{3}\boldsymbol{w}$. The combinations $c\boldsymbol{v} + c\boldsymbol{w}$ fill out what line?

18 Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

19 Restricted only by $c \ge 0$ and $d \ge 0$ draw the "cone" of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

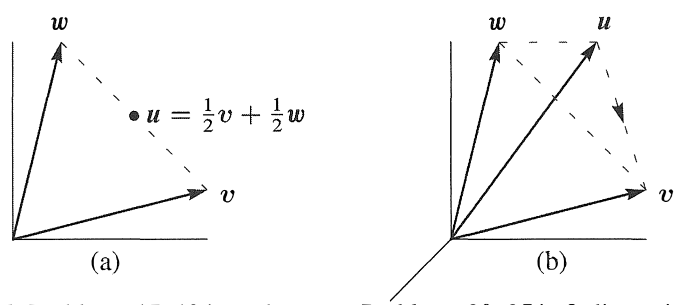


Figure 1.5: Problems **15–19** in a plane   Problems **20–25** in 3-dimensional space

Problems 20–25 deal with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in three-dimensional space (see Figure 1.5b).

20 Locate $\frac{1}{3}\boldsymbol{u}+\frac{1}{3}\boldsymbol{v}+\frac{1}{3}\boldsymbol{w}$ and $\frac{1}{2}\boldsymbol{u}+\frac{1}{2}\boldsymbol{w}$ in Figure 1.5b. Challenge problem: Under what restrictions on c, d, e, will the combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ fill in the dashed triangle? To stay in the triangle, one requirement is $c \geq 0, d \geq 0, e \geq 0$.

21 The three sides of the dashed triangle are $\boldsymbol{v}-\boldsymbol{u}$ and $\boldsymbol{w}-\boldsymbol{v}$ and $\boldsymbol{u}-\boldsymbol{w}$. Their sum is _____. Draw the head-to-tail addition around a plane triangle of $(3,1)$ plus $(-1,1)$ plus $(-2,-2)$.

22 Shade in the pyramid of combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ with $c \geq 0$, $d \geq 0$, $e \geq 0$ and $c+d+e \leq 1$. Mark the vector $\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$ as inside or outside this pyramid.

23 If you look at *all* combinations of those $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$, is there any vector that can't be produced from $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$? Different answer if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in _____.

24 Which vectors are combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$, and *also* combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$?

25 Draw vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ so that their combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ fill only a line. Find vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ so that their combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ fill only a plane.

26 What combination $c\begin{bmatrix}1\\2\end{bmatrix}+d\begin{bmatrix}3\\1\end{bmatrix}$ produces $\begin{bmatrix}14\\8\end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

Challenge Problems

27 How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges? A typical corner is $(0,0,1,0)$. A typical edge goes to $(0,1,0,0)$.

28 Find vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ so that $\boldsymbol{v}+\boldsymbol{w}=(4,5,6)$ and $\boldsymbol{v}-\boldsymbol{w}=(2,5,8)$. This is a question with _____ unknown numbers, and an equal number of equations to find those numbers.

29 Find *two different combinations* of the three vectors $\boldsymbol{u}=(1,3)$ and $\boldsymbol{v}=(2,7)$ and $\boldsymbol{w}=(1,5)$ that produce $\boldsymbol{b}=(0,1)$. Slightly delicate question: If I take any three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in the plane, will there always be two different combinations that produce $\boldsymbol{b}=(0,1)$?

30 The linear combinations of $\boldsymbol{v}=(a,b)$ and $\boldsymbol{w}=(c,d)$ fill the plane unless _____. Find four vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{z}$ with four components each so that their combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}+f\boldsymbol{z}$ produce all vectors (b_1, b_2, b_3, b_4) in four-dimensional space.

31 Write down three equations for c, d, e so that $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}=\boldsymbol{b}$. Can you somehow find c, d, e for this $\boldsymbol{b}$?

$$\boldsymbol{u}=\begin{bmatrix}2\\-1\\0\end{bmatrix} \quad \boldsymbol{v}=\begin{bmatrix}-1\\2\\-1\end{bmatrix} \quad \boldsymbol{w}=\begin{bmatrix}0\\-1\\2\end{bmatrix} \quad \boldsymbol{b}=\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

1.2 Lengths and Dot Products

1 The "dot product" of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is $\boldsymbol{v} \cdot \boldsymbol{w} = (1)(4) + (2)(5) = 4 + 10 = \mathbf{14}$.

2 $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$ are perpendicular because $\boldsymbol{v} \cdot \boldsymbol{w}$ is zero: $(1)(4) + (3)(-4) + (2)(4) = \mathbf{0}$.

3 The length squared of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is $\boldsymbol{v} \cdot \boldsymbol{v} = 1 + 9 + 4 = 14$. **The length is** $||\boldsymbol{v}|| = \sqrt{\mathbf{14}}$.

4 Then $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{||\boldsymbol{v}||} = \dfrac{\boldsymbol{v}}{\sqrt{14}} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ has length $||\boldsymbol{u}|| = \mathbf{1}$. Check $\dfrac{1}{14} + \dfrac{9}{14} + \dfrac{4}{14} = 1$.

5 The angle θ between $\boldsymbol{v}$ and $\boldsymbol{w}$ has $\cos\theta = \dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}|| \; ||\boldsymbol{w}||}$.

6 The angle between $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $\cos\theta = \dfrac{1}{(1)(\sqrt{2})}$. That angle is $\theta = 45°$.

7 All angles have $|\cos\theta| \leq 1$. So all vectors have $\boxed{|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}|| \; ||\boldsymbol{w}||.}$

The first section backed off from multiplying vectors. Now we go forward to define the *"dot product"* of $\boldsymbol{v}$ and $\boldsymbol{w}$. This multiplication involves the separate products $v_1 w_1$ and $v_2 w_2$, but it doesn't stop there. Those two numbers are added to produce one number $\boldsymbol{v} \cdot \boldsymbol{w}$.

This is the geometry section (*lengths of vectors and cosines of angles between them*).

The ***dot product*** or ***inner product*** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2. \tag{1}$$

Example 1 The vectors $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ have a *zero* dot product:

Dot product is zero
Perpendicular vectors

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular*. The angle between them is 90°. When we drew them in Figure 1.1, we saw a rectangle (not just any parallelogram). The clearest example of perpendicular vectors is $\boldsymbol{i} = (1, 0)$ along the x axis and $\boldsymbol{j} = (0, 1)$ up the y axis. Again the dot product is $\boldsymbol{i} \cdot \boldsymbol{j} = 0 + 0 = 0$. Those vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ form a right angle.

The dot product of $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (3, 1)$ is 5. Soon $\boldsymbol{v} \cdot \boldsymbol{w}$ will reveal the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ (not 90°). Please check that $\boldsymbol{w} \cdot \boldsymbol{v}$ is also 5.

The dot product $\boldsymbol{w} \cdot \boldsymbol{v}$ equals $\boldsymbol{v} \cdot \boldsymbol{w}$. The order of $\boldsymbol{v}$ and $\boldsymbol{w}$ makes no difference.

Example 2 Put a weight of 4 at the point $x = -1$ (left of zero) and a weight of 2 at the point $x = 2$ (right of zero). The x axis will balance on the center point (like a see-saw). The weights balance because the dot product is $(4)(-1) + (2)(2) = 0$.

This example is typical of engineering and science. The vector of weights is $(w_1, w_2) = (4, 2)$. The vector of distances from the center is $(v_1, v_2) = (-1, 2)$. The weights times the distances, $w_1 v_1$ and $w_2 v_2$, give the "moments". The equation for the see-saw to balance is $w_1 v_1 + w_2 v_2 = 0$.

Example 3 Dot products enter in economics and business. We have three goods to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are (q_1, q_2, q_3)—positive when we sell, negative when we buy. *Selling q_1 units at the price p_1 brings in $q_1 p_1$.* The total income (quantities q times prices p) is ***the dot product $\boldsymbol{q} \cdot \boldsymbol{p}$ in three dimensions***:

$$\textbf{\textit{Income}} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3 = \textbf{\textit{dot product}}.$$

A zero dot product means that "the books balance". Total sales equal total purchases if $\boldsymbol{q} \cdot \boldsymbol{p} = 0$. Then $\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$ (in three-dimensional space). A supermarket with thousands of goods goes quickly into high dimensions.

Small note: Spreadsheets have become essential in management. They compute linear combinations and dot products. What you see on the screen is a matrix.

Main point For $\boldsymbol{v} \cdot \boldsymbol{w}$, multiply each v_i times w_i. Then $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + \cdots + v_n w_n$.

Lengths and Unit Vectors

An important case is the dot product of a vector *with itself.* In this case $\boldsymbol{v}$ equals $\boldsymbol{w}$. When the vector is $\boldsymbol{v} = (1, 2, 3)$, the dot product with itself is $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2 = 14$:

Dot product $\boldsymbol{v} \cdot \boldsymbol{v}$
Length squared

$$\|\boldsymbol{v}\|^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = \mathbf{14}.$$

Instead of a 90° angle between vectors we have 0°. The answer is not zero because $\boldsymbol{v}$ is not perpendicular to itself. The dot product $\boldsymbol{v} \cdot \boldsymbol{v}$ gives the *length of $\boldsymbol{v}$ squared.*

DEFINITION The ***length*** $\|\boldsymbol{v}\|$ of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$:

$$\textbf{length} = \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)^{1/2}.$$

In two dimensions the length is $\sqrt{v_1^2 + v_2^2}$. In three dimensions it is $\sqrt{v_1^2 + v_2^2 + v_3^2}$. By the calculation above, the length of $\boldsymbol{v} = (1, 2, 3)$ is $\|\boldsymbol{v}\| = \sqrt{14}$.

Here $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ is just the ordinary length of the arrow that represents the vector. If the components are 1 and 2, the arrow is the third side of a right triangle (Figure 1.6). The Pythagoras formula $a^2 + b^2 = c^2$ connects the three sides: $1^2 + 2^2 = \|\boldsymbol{v}\|^2$.

For the length of $\boldsymbol{v} = (1, 2, 3)$, we used the right triangle formula twice. The vector $(1, 2, 0)$ in the base has length $\sqrt{5}$. This base vector is perpendicular to $(0, 0, 3)$ that goes straight up. So the diagonal of the box has length $\|\boldsymbol{v}\| = \sqrt{5 + 9} = \sqrt{14}$.

The length of a four-dimensional vector would be $\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$. Thus the vector $(1, 1, 1, 1)$ has length $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. That diagonal in n dimensions has length $\sqrt{n}$.

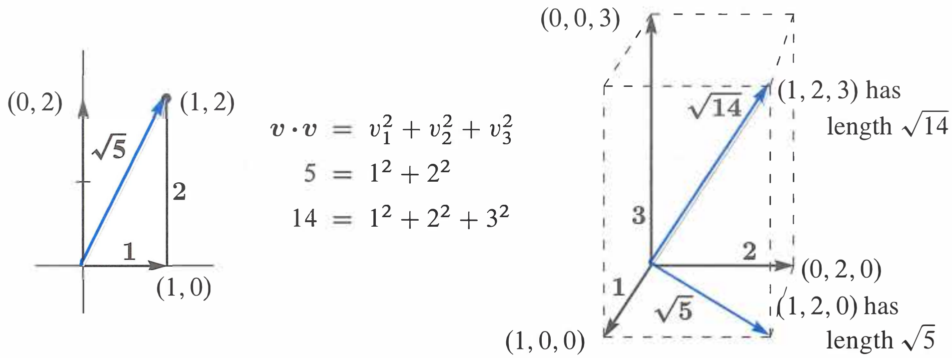


Figure 1.6: The length $\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ of two-dimensional and three-dimensional vectors.

The word "**unit**" is always indicating that some measurement equals "one". The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we see the meaning of a "unit vector".

DEFINITION ***A unit vector $\boldsymbol{u}$ is a vector whose length equals one***. Then $\boldsymbol{u} \cdot \boldsymbol{u} = 1$.

An example in four dimensions is $\boldsymbol{u} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Then $\boldsymbol{u} \cdot \boldsymbol{u}$ is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. We divided $\boldsymbol{v} = (1, 1, 1, 1)$ by its length $\|\boldsymbol{v}\| = 2$ to get this unit vector.

Example 4 The standard unit vectors along the x and y axes are written $\boldsymbol{i}$ and $\boldsymbol{j}$. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $(\cos\theta, \sin\theta)$:

$$\textbf{Unit vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

When $\theta = 0$, the horizontal vector $\boldsymbol{u}$ is $\boldsymbol{i}$. When $\theta = 90°$ (or $\frac{\pi}{2}$ radians), the vertical vector is $\boldsymbol{j}$. At any angle, the components $\cos\theta$ and $\sin\theta$ produce $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ because

$\cos^2\theta + \sin^2\theta = 1$. These vectors reach out to the unit circle in Figure 1.7. Thus $\cos\theta$ and $\sin\theta$ are simply the coordinates of that point at angle θ on the unit circle.

Since $(2, 2, 1)$ has length 3, the vector $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ has length 1. Check that $\boldsymbol{u} \cdot \boldsymbol{u} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$. For a unit vector, **divide any nonzero vector $\boldsymbol{v}$ by its length $\|\boldsymbol{v}\|$**.

Unit vector $\boldsymbol{u} = \boldsymbol{v} / \|\boldsymbol{v}\|$ **is a unit vector in the same direction as $\boldsymbol{v}$.**

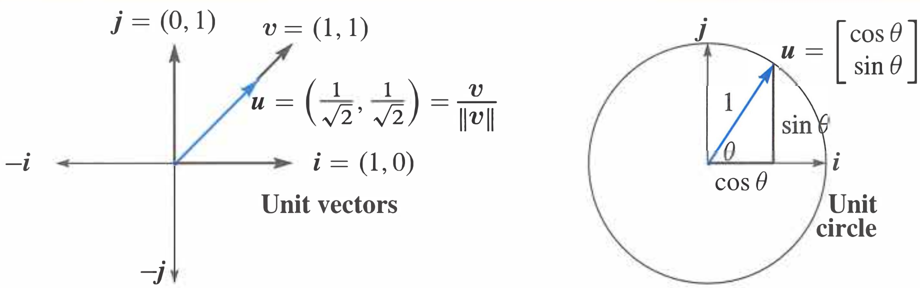


Figure 1.7: The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ at angle $45°$ (left) divides $\boldsymbol{v} = (1, 1)$ by its length $\|\boldsymbol{v}\| = \sqrt{2}$. The unit vector $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is at angle θ.

The Angle Between Two Vectors

We stated that perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is $90°$. To explain this, we have to connect angles to dot products. Then we show how $\boldsymbol{v} \cdot \boldsymbol{w}$ finds the angle between any two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

Right angles ***The dot product is $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ when $\boldsymbol{v}$ is perpendicular to $\boldsymbol{w}$.***

Proof When $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, they form two sides of a right triangle. The third side is $\boldsymbol{v} - \boldsymbol{w}$ (the hypotenuse going across in Figure 1.8). The *Pythagoras Law* for the sides of a right triangle is $a^2 + b^2 = c^2$:

$$\textbf{Perpendicular vectors} \qquad \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2 \qquad (2)$$

Writing out the formulas for those lengths in two dimensions, this equation is

$$\textbf{Pythagoras} \qquad (v_1^2 + v_2^2) + (w_1^2 + w_2^2) = (v_1 - w_1)^2 + (v_2 - w_2)^2. \qquad (3)$$

The right side begins with $v_1^2 - 2v_1w_1 + w_1^2$. Then v_1^2 and w_1^2 are on both sides of the equation and they cancel, leaving $-2v_1w_1$. Also v_2^2 and w_2^2 cancel, leaving $-2v_2w_2$. (In three dimensions there would be $-2v_3w_3$.) Now divide by -2 to see $\boldsymbol{v} - \boldsymbol{w} = 0$:

$$0 = -2v_1w_1 - 2v_2w_2 \quad \text{which leads to} \quad \boldsymbol{v_1w_1 + v_2w_2 = 0}. \qquad (4)$$

Conclusion Right angles produce $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is $\theta = 90°$. Then $\cos\theta = 0$. The zero vector $\boldsymbol{v} = \mathbf{0}$ is perpendicular to every vector $\boldsymbol{w}$ because $\mathbf{0} \cdot \boldsymbol{w}$ is always zero.

Now suppose $\boldsymbol{v} \cdot \boldsymbol{w}$ is **not zero**. It may be positive, it may be negative. The sign of $\boldsymbol{v} \cdot \boldsymbol{w}$ immediately tells whether we are below or above a right angle. The angle is less than $90°$ when $\boldsymbol{v} \cdot \boldsymbol{w}$ is positive. The angle is above $90°$ when $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative. The right side of Figure 1.8 shows a typical vector $\boldsymbol{v} = (3, 1)$. The angle with $\boldsymbol{w} = (1, 3)$ is less than $90°$ because $\boldsymbol{v} \cdot \boldsymbol{w} = 6$ is positive.

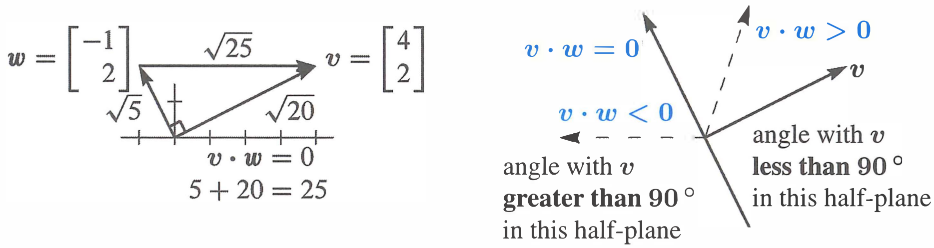


Figure 1.8: Perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. Then $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2$.

The borderline is where vectors are perpendicular to $\boldsymbol{v}$. On that dividing line between plus and minus, $(1, -3)$ is perpendicular to $(3, 1)$. The dot product is zero.

The dot product reveals the exact angle θ. For unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$, the sign of $\boldsymbol{u} \cdot \boldsymbol{U}$ tells whether $\theta < 90°$ or $\theta > 90°$. More than that, ***the dot product $\boldsymbol{u} \cdot \boldsymbol{U}$ is the cosine of θ.*** This remains true in n dimensions.

Unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ at angle θ have $\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta$. **Certainly $|\boldsymbol{u} \cdot \boldsymbol{U}| \leq 1$.**

Remember that $\cos\theta$ is never greater than 1. It is never less than -1. *The dot product of unit vectors is between -1 and 1.* **The cosine of θ is revealed by $\boldsymbol{u} \cdot \boldsymbol{U}$.**

Figure 1.9 shows this clearly when the vectors are $\boldsymbol{u} = (\cos\theta, \sin\theta)$ and $\boldsymbol{i} = (1, 0)$. The dot product is $\boldsymbol{u} \cdot \boldsymbol{i} = \cos\theta$. That is the cosine of the angle between them.

After rotation through any angle α, these are still unit vectors. The vector $\boldsymbol{i} = (1, 0)$ rotates to $(\cos\alpha, \sin\alpha)$. The vector $\boldsymbol{u}$ rotates to $(\cos\beta, \sin\beta)$ with $\beta = \alpha + \theta$. Their dot product is $\cos\alpha\cos\beta + \sin\alpha\sin\beta$. From trigonometry this is $\cos(\beta - \alpha) = \cos\theta$.

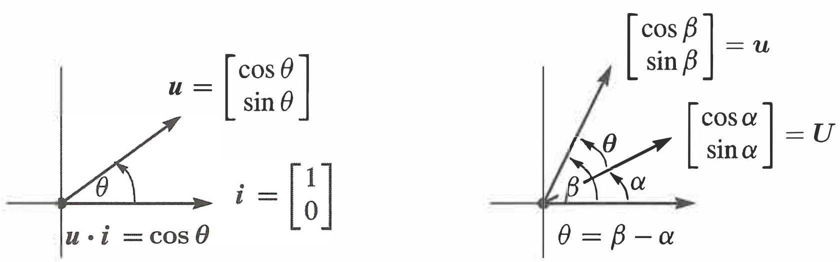


Figure 1.9: Unit vectors: $\boldsymbol{u} \cdot \boldsymbol{U}$ is the cosine of θ (the angle between).

What if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not unit vectors? Divide by their lengths to get $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ and $\boldsymbol{U} = \boldsymbol{w}/\|\boldsymbol{w}\|$. Then the dot product of those unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ gives $\cos\theta$.

COSINE FORMULA If $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors then $\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \cos\theta.$ (5)

Whatever the angle, this dot product of $\boldsymbol{v}/\|\boldsymbol{v}\|$ with $\boldsymbol{w}/\|\boldsymbol{w}\|$ never exceeds one. That is the ***"Schwarz inequality"*** $|\boldsymbol{v}\cdot\boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ for dot products—or more correctly the Cauchy-Schwarz-Buniakowsky inequality. It was found in France and Germany and Russia (and maybe elsewhere—it is the most important inequality in mathematics).

Since $|\cos\theta|$ never exceeds 1, the cosine formula gives two great inequalities:

SCHWARZ INEQUALITY	$\|\boldsymbol{v}\cdot\boldsymbol{w}\| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$
TRIANGLE INEQUALITY	$\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

Example 5 Find $\cos\theta$ for $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $\boldsymbol{v}\cdot\boldsymbol{w} = 4$. Both $\boldsymbol{v}$ and $\boldsymbol{w}$ have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

By the Schwarz inequality, $\boldsymbol{v}\cdot\boldsymbol{w} = 4$ is less than $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\| = 5$. By the triangle inequality, side $3 = \|\boldsymbol{v}+\boldsymbol{w}\|$ is less than side 1 + side 2. For $\boldsymbol{v}+\boldsymbol{w} = (3,3)$ the three sides are $\sqrt{18} < \sqrt{5}+\sqrt{5}$. Square this triangle inequality to get $18 < 20$.

Example 6 The dot product of $\boldsymbol{v} = (a,b)$ and $\boldsymbol{w} = (b,a)$ is $2ab$. Both lengths are $\sqrt{a^2+b^2}$. The Schwarz inequality $\boldsymbol{v}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ says that $\boldsymbol{2ab \le a^2 + b^2}$.

This is more famous if we write $x = a^2$ and $y = b^2$. The "geometric mean" $\sqrt{\boldsymbol{xy}}$ is not larger than the "arithmetic mean" = average $\frac{1}{2}(\boldsymbol{x}+\boldsymbol{y})$.

$$\begin{array}{c}\textbf{Geometric}\\\textbf{mean}\end{array} \le \begin{array}{c}\textbf{Arithmetic}\\\textbf{mean}\end{array} \qquad ab \le \frac{a^2+b^2}{2} \quad \text{becomes} \quad \sqrt{xy} \le \frac{x+y}{2}.$$

Example 5 had $a = 2$ and $b = 1$. So $x = 4$ and $y = 1$. The geometric mean $\sqrt{xy} = 2$ is below the arithmetic mean $\frac{1}{2}(1+4) = 2.5$.

Notes on Computing

MATLAB, Python *and* Julia *work directly with whole vectors, not their components.* When $\boldsymbol{v}$ and $\boldsymbol{w}$ have been defined, $\boldsymbol{v}+\boldsymbol{w}$ is immediately understood. Input $\boldsymbol{v}$ and $\boldsymbol{w}$ as rows—the prime $'$ transposes them to columns. $2\boldsymbol{v}+3\boldsymbol{w}$ becomes $2*\boldsymbol{v}+3*\boldsymbol{w}$. The result will be printed unless the line ends in a semicolon.

MATLAB $\quad v = [2 \quad 3 \quad 4]' \;\; ; \quad w = [1 \quad 1 \quad 1]' \;\; ; \quad u = 2 * v + 3 * w$

The dot product $v \cdot w$ is ***a row vector times a column vector*** (*use* * *instead of* $\cdot$):

Instead of	$\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}3\\4\end{bmatrix}$	we more often see $[1 \;\; 2]\begin{bmatrix}3\\4\end{bmatrix}$	or	$v' * w$

The length of v is known to MATLAB as norm (v). This is sqrt $(v' * v)$. Then find the cosine from the dot product $v' * w$ and the angle (in radians) that has that cosine:

Cosine formula	cosine $= v' * w/(\text{norm}\,(v) * \text{norm}\,(w))$
The arc cosine	angle $=$ acos (cosine)

An M-file would create a new function **cosine** (v, w). Python and Julia are open source.

■ REVIEW OF THE KEY IDEAS ■

1. The dot product $v \cdot w$ multiplies each component v_i by w_i and adds all $v_i w_i$.
2. The length $\|v\|$ is the square root of $v \cdot v$. Then $u = v/\|v\|$ is a ***unit vector***: length 1.
3. The dot product is $v \cdot w = 0$ when vectors v and w are perpendicular.
4. The cosine of θ (the angle between any nonzero v and w) never exceeds 1:

$$\textbf{Cosine} \quad \cos\theta = \frac{v \cdot w}{\|v\|\|w\|} \qquad \textit{Schwarz inequality} \quad |v \cdot w| \leq \|v\| \, \|w\|.$$

■ WORKED EXAMPLES ■

1.2 A For the vectors $v = (3, 4)$ and $w = (4, 3)$ test the Schwarz inequality on $v \cdot w$ and the triangle inequality on $\|v + w\|$. Find $\cos\theta$ for the angle between v and w. Which v and w give *equality* $|v \cdot w| = \|v\| \, \|w\|$ and $\|v + w\| = \|v\| + \|w\|$?

Solution The dot product is $v \cdot w = (3)(4) + (4)(3) = 24$. The length of v is $\|v\| = \sqrt{9 + 16} = 5$ and also $\|w\| = 5$. The sum $v + w = (7, 7)$ has length $7\sqrt{2} < 10$.

Schwarz inequality	$\lvert v \cdot w\rvert \leq \|v\| \, \|w\|$	is	$24 < 25$.
Triangle inequality	$\|v + w\| \leq \|v\| + \|w\|$	is	$7\sqrt{2} < 5 + 5$.
Cosine of angle	$\cos\theta = \frac{24}{25}$	Thin angle from $v = (3, 4)$ to $w = (4, 3)$	

Equality: One vector is a multiple of the other as in $w = cv$. Then the angle is $0°$ or $180°$. In this case $|\cos\theta| = 1$ and $|v \cdot w|$ *equals* $\|v\| \, \|w\|$. If the angle is $0°$, as in $w = 2v$, then $\|v + w\| = \|v\| + \|w\|$ (both sides give $3\|v\|$). This $v, 2v, 3v$ triangle is flat!

1.2 B Find a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = (3,4)$. Find a unit vector $\boldsymbol{U}$ that is perpendicular to $\boldsymbol{u}$. How many possibilities for $\boldsymbol{U}$?

Solution For a unit vector $\boldsymbol{u}$, divide $\boldsymbol{v}$ by its length $\|\boldsymbol{v}\| = 5$. For a perpendicular vector $\boldsymbol{V}$ we can choose $(-4,3)$ since the dot product $\boldsymbol{v} \cdot \boldsymbol{V}$ is $(3)(-4) + (4)(3) = 0$. For a *unit* vector perpendicular to $\boldsymbol{u}$, divide $\boldsymbol{V}$ by its length $\|\boldsymbol{V}\|$:

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \boldsymbol{U} = \frac{\boldsymbol{V}}{\|\boldsymbol{V}\|} = \left(-\frac{4}{5}, \frac{3}{5}\right) \qquad \boldsymbol{u} \cdot \boldsymbol{U} = 0$$

The only other perpendicular unit vector would be $-\boldsymbol{U} = (\frac{4}{5}, -\frac{3}{5})$.

1.2 C Find a vector $\boldsymbol{x} = (c,d)$ that has dot products $\boldsymbol{x} \cdot \boldsymbol{r} = 1$ and $\boldsymbol{x} \cdot \boldsymbol{s} = 0$ with two given vectors $\boldsymbol{r} = (2,-1)$ and $\boldsymbol{s} = (-1,2)$.

Solution Those two dot products give linear equations for c and d. Then $\boldsymbol{x} = (c,d)$.

$$\begin{array}{lll} \boldsymbol{x} \cdot \boldsymbol{r} = 1 & \text{is} & 2c - d = 1 \\ \boldsymbol{x} \cdot \boldsymbol{s} = 0 & \text{is} & -c + 2d = 0 \end{array}$$

The same equations as in Worked Example 1.1 C

Comment on n equations for $\boldsymbol{x} = (x_1, \ldots, x_n)$ in n-dimensional space

Section 1.1 would start with columns $\boldsymbol{v}_j$. The goal is to produce $x_1\boldsymbol{v}_1 + \cdots + x_n\boldsymbol{v}_n = \boldsymbol{b}$. This section would start from rows $\boldsymbol{r}_i$. Now the goal is to find $\boldsymbol{x}$ with $\boldsymbol{x} \cdot \boldsymbol{r}_i = b_i$.

Soon the $\boldsymbol{v}$'s will be the columns of a matrix A, and the $\boldsymbol{r}$'s will be the rows of A. Then the (one and only) problem will be to solve $A\boldsymbol{x} = \boldsymbol{b}$.

Problem Set 1.2

1 Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:

$$\boldsymbol{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2 Compute the lengths $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ and $\|\boldsymbol{w}\|$ of those vectors. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \le \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$.

3 Find unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$ in Problem 1, and the cosine of the angle θ. Choose vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ that make $0°$, $90°$, and $180°$ angles with $\boldsymbol{w}$.

4 For any *unit* vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, find the dot products (actual numbers) of

(a) $\boldsymbol{v}$ and $-\boldsymbol{v}$ (b) $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ (c) $\boldsymbol{v} - 2\boldsymbol{w}$ and $\boldsymbol{v} + 2\boldsymbol{w}$

5 Find unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in the directions of $\boldsymbol{v} = (1,3)$ and $\boldsymbol{w} = (2,1,2)$. Find unit vectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ that are perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 (a) Describe every vector $\boldsymbol{w} = (w_1, w_2)$ that is perpendicular to $\boldsymbol{v} = (2, -1)$.

(b) All vectors perpendicular to $V = (1, 1, 1)$ lie on a _____ in 3 dimensions.

(c) The vectors perpendicular to both $(1, 1, 1)$ and $(1, 2, 3)$ lie on a _____.

7 Find the angle θ (from its cosine) between these pairs of vectors:

(a) $\boldsymbol{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\boldsymbol{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(c) $\boldsymbol{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ (d) $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

8 True or false (give a reason if true or find a counterexample if false):

(a) If $\boldsymbol{u} = (1, 1, 1)$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{v}$ is parallel to $\boldsymbol{w}$.

(b) If $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v} + 2\boldsymbol{w}$.

(c) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular unit vectors then $||\boldsymbol{u} - \boldsymbol{v}|| = \sqrt{2}$. *Yes*!

9 The slopes of the arrows from $(0, 0)$ to (v_1, v_2) and (w_1, w_2) are v_2/v_1 and w_2/w_1. **Suppose the product** v_2w_2/v_1w_1 **of those slopes is** -1. Show that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the vectors are perpendicular. (The line $y = 4x$ is perpendicular to $y = -\frac{1}{4}x$.)

10 Draw arrows from $(0, 0)$ to the points $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (-2, 1)$. Multiply their slopes. That answer is a signal that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the arrows are _____.

11 If $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative, what does this say about the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$? Draw a 3-dimensional vector $\boldsymbol{v}$ (an arrow), and show where to find all $\boldsymbol{w}$'s with $\boldsymbol{v} \cdot \boldsymbol{w} < 0$.

12 With $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (1, 5)$ choose a number c so that $\boldsymbol{w} - c\boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$. Then find the formula for c starting from *any* nonzero $\boldsymbol{v}$ and $\boldsymbol{w}$.

13 Find nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that are perpendicular to $(1, 0, 1)$ and to each other.

14 Find nonzero vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that are perpendicular to $(1, 1, 1, 1)$ and to each other.

15 The geometric mean of $x = 2$ and $y = 8$ is $\sqrt{xy} = 4$. The arithmetic mean is larger: $\frac{1}{2}(x + y) =$ _____. This would come in Example 6 from the Schwarz inequality for $\boldsymbol{v} = (\sqrt{2}, \sqrt{8})$ and $\boldsymbol{w} = (\sqrt{8}, \sqrt{2})$. Find $\cos\theta$ for this $\boldsymbol{v}$ and $\boldsymbol{w}$.

16 **How long is the vector** $\boldsymbol{v} = (1, 1, \ldots, 1)$ **in** 9 **dimensions**? Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$ and a unit vector $\boldsymbol{w}$ that is perpendicular to $\boldsymbol{v}$.

17 What are the cosines of the angles α, β, θ between the vector $(1, 0, -1)$ and the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ along the axes? Check the formula $\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$.